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A SUGGESTED MODIFICATION OF COMPUTATIONAL PROCEDURES FOR SPHERICAL COORDINATES

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Introduction

By way of introduction, this note is concerned with finite-difference meteorological equations in spherical coordinates and a net regular in latitude and longitude.

Certain kinds of terms repeatedly crop up, for example, in the two equations of motion

$$\frac{u}{a\cos\Phi}\left(\frac{\partial u}{\partial\lambda} - v\sin\Phi\right) \tag{1a}$$

$$\frac{u}{a\cos\Phi}\left(\frac{\partial v}{\partial\lambda} + u\sin\Phi\right) \tag{1b}$$

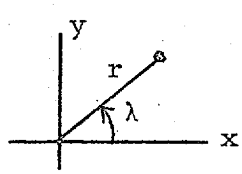
and in the PE barotropic equation,

$$\frac{1}{a\cos\Phi}\left(\frac{\partial hu}{\partial\lambda} - hv\sin\Phi\right) \tag{1c}$$

where a is radius of the earth, Φ north latitude, λ east longitude, u eastward velocity component, v northward velocity component, and h height of the free surface.

Now, the co-latitude, $\theta = \frac{1}{2}\pi - \Phi$, $\cos\Phi = \sin\theta$, and $\sin\Phi = \cos\theta$. Therefore, $\sin\Phi = 1 - \frac{1}{2}\theta^2 + (1/24)\,\theta^4 - \cdots$, and as a first-order approximation near the North Pole $\sin\Phi$ is unity. Similarly, $\cos\Phi \cong \theta$ near the North Pole. This justifies study of the problems near the pole, with polar coordinates on a plane tangent at the pole, and neglect of variation from unity of any map-scale factor.

In the figure, the origin is at the pole. Where U is the x-component of velocity, and V the y-component,



$$U = -u\sin\lambda - v\cos\lambda \tag{2}$$

$$V = u\cos\lambda - v\sin\lambda$$

or

$$u = -U\sin\lambda + V\cos\lambda \tag{3}$$

$$v = -U\cos\lambda - V\sin\lambda$$

On the tangent plane, (1) become

$$\frac{u}{r}\left(\frac{\partial u}{\partial \lambda} - v\right) \tag{4a}$$

$$\frac{u}{r}\left(\frac{\partial v}{\partial \lambda} + u\right) \tag{4b}$$

$$\frac{1}{r}\left(\frac{\partial h u}{\partial \lambda} - h v\right) \tag{4c}$$

Substituting from (3) into (4a), we find

$$\frac{u}{r}\left(-\frac{\partial U}{\partial \lambda}\sin\lambda + \frac{\partial V}{\partial \lambda}\cos\lambda\right) \tag{5}$$

and, since U and V have no special behavior in physical space near the pole,

$$\lim_{r \to 0} \frac{\partial U}{\partial \lambda} = \lim_{r \to 0} \frac{\partial V}{\partial \lambda} = 0$$

Or, to put it another way, if we consider the difference of U or V between two points separated by a very small difference, a few degrees, in λ, then the difference of U or V approaches zero as the pole is approached because the two points approach each other if the difference in λ is held constant. We conclude, therefore, that

$$\lim_{r \to 0} \left(\frac{\partial u}{\partial \lambda} - v\right) = 0$$

and from similar arguments, the corresponding factors of (4b) and (4c) also vanish at the pole.

Note, by the way, that $\partial u/\partial \lambda$ and v individually do not vanish at the pole:

$$\lim_{r \to 0} \frac{\partial u}{\partial \lambda} = -U\cos\lambda - V\sin\lambda = v$$

Each of (4), then formally becomes, as r approaches zero, the indeterminate 0/0. The indeterminancy is resolved, however, when we note in (5) that

$$\frac{1}{r}\frac{\partial U}{\partial \lambda} = \frac{\partial U}{\partial s}\ ; \ \frac{1}{r}\frac{\partial V}{\partial \lambda} = \frac{\partial V}{\partial s}$$

where s is distance along a latitude circle. These derivatives have perfectly good non-zero finite values,generally, and therefore so does (5), no matter how close to the pole we are. For a unique value at the pole, however, a value of λ must be associated with (5).

Now, consider a finite-difference approximation to (4a):

$$(\bar{r}^{r})^{-1}\ \overline{\bar{u}^{\lambda}(u_{\lambda} - \bar{v}^{\lambda})}^{r} \tag{6}$$

which is an estimate in a grid box bounded by two grid meridians and two grid latitude circles. In the case of the northernmost tier of grid boxes, the northern bounding latitude circle collapes to the pole point.

Parenthetically, the quantity $u_{\lambda} - \bar{v}^{\lambda}$ will not be exactly zero, as in the case of its differential analogue, but will be very small if $\Delta\lambda$ is only a few degrees. A good procedure would be to use zero there rather than a calculated erroneous value, or one could reduce the calculated value to exactly zero at the pole by including a factor on one or both of the terms depending only on $\Delta\lambda$ and which would very nearly be unity.

Procedures in the numerical systems under discussion generally do not yield at the poles a single velocity vector, nor a single h in a barotropic model, unless a procedure is deliberately included like averaging all values at the pole. My thesis here is that <u>the pole points are not unique in this respect</u>, but only exhibit an extreme of a problem that should be expected throughout polar areas. In (6) in a box adjacent to the pole the factor $u_{\lambda} - \bar{v}^{\lambda}$ will be vanishingly small at the pole only if u and v at each pole point, associated with different longitude angles, represent the same vector.

A similar situation exists along the other grid latitude circles and only to a somewhat lesser degree. Furthermore, if u and v during an integration become noisy, even moderately so, $u_\lambda - \bar{v}^\lambda$ will not, as it should, pass more or less uniformly to zero as we step poleward from grid point to grid point near the pole. Then when we divide by $\bar{r}^r$, a small number near the pole, we will get not only erroneously noisy, but also too large, values for (6).

The modification which I suggest is

$$(\bar{r}^r)^{-1} \overline{\overline{\bar{u}^\lambda (u_\lambda - \bar{v}^\lambda)}^{r\lambda}} \qquad (7)$$

The additional $\overline{(\ )}^\lambda$ in (7), compared with (6), will serve to suppress spatial high frequencies in the tendencies. Alternatively, a term could be added to the equations to damp spatial high frequencies in the stored fields, e. g., the term $-K^2 u_{\lambda\lambda}$ in the u-equation. The coefficient, K^2, should vary strongly with latitude, large near the pole and perhaps vanishing at the equator.

Note that development of spatial high-frequency noise, in an integration, according to Robert, Shuman, and Gerrity, would be accompanied by exponential instability.